

Off-diagonal long-range order, cycle probabilities, and condensate fraction in the ideal Bose gas

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We discuss the relationship between the cycle probabilities in the path-integral representation of the ideal Bose gas, off-diagonal long-range order, and Bose-Einstein condensation. Starting from the Landsberg recursion relation for the canonic partition function, we use elementary considerations to show that in a box of size L^3 the sum of the cycle probabilities of length $k \gg L^2$ equals the off-diagonal long-range order parameter in the thermodynamic limit. For arbitrary systems of ideal bosons, the integer derivative of the cycle probabilities is related to the probability of condensing k bosons. We use this relation to derive the precise form of the π_k in the thermodynamic limit. We also determine the function π_k for arbitrary systems. Furthermore, we use the cycle probabilities to compute the probability distribution of the maximum-length cycles both at $T=0$, where the ideal Bose gas reduces to the study of random permutations, and at finite temperature. We close with comments on the cycle probabilities in interacting Bose gases.

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I. INTRODUCTION

The canonic partition function of N ideal bosons in a system with energy levels $\epsilon_0=0 < \epsilon_1 < \dots$ at inverse temperature $\beta=1/(k_B T)$ is given by

$$Z_N = \left[\prod_{\sigma} \sum_{n_{\sigma}=0}^N \right] \exp\left(-\beta \sum_{\sigma} n_{\sigma} \epsilon_{\sigma}\right) \delta_{\sum_{\sigma} n_{\sigma}, N}. \quad (1)$$

For each state σ , the allowed occupation numbers n_{σ} go from 0 to N , and the Kronecker δ function enforces the total number of particles to be N .

The Feynman path-integral [1] represents Z_N as a trace over the diagonal density matrix ρ^{bos} ,

$$Z_N = \int dx_1 \cdots dx_N \rho^{\text{bos}}[\{x_1, \dots, x_N\}, \{x_1, \dots, x_N\}, \beta],$$

which is given in terms of the $N!$ permutations P of the distinguishable-particle density matrix ρ ,

$$\begin{aligned} \rho^{\text{bos}}[\{x_1, \dots, x_N\}, \{x'_1, \dots, x'_N\}, \beta] \\ = \frac{1}{N!} \sum_P \rho[\{x_1, \dots, x_N\}, \{x'_{P_1}, \dots, x'_{P_N}\}, \beta]. \end{aligned} \quad (2)$$

Thus the partition function is a sum of $N!$ permutation-dependent terms, $Z_N = \sum_P Z_P$, and it is possible to define the probability of a permutation P as

$$\pi_P = \frac{Z_P}{Z_N}. \quad (3)$$

Permutations can be broken up into cycles, and one may also define cycle probabilities:

$$\pi_k = \frac{1}{Z_N} \sum_{P \in \{N, k\}} Z_P, \quad (4)$$

where $\{N, k\}$ denotes the set of all permutations where the particle N belongs to a cycle of length k . The choice of the particle N in Eq. (4) is arbitrary.

In this paper, we are concerned with the characterization of the cycle probabilities π_k in the ideal Bose gas and with their relation to quantities characterizing Bose-Einstein condensation, namely the off-diagonal long-range order (in Sec. III, see also Refs. [2,3]) and the condensate fraction (in Sec. IV). In fact, for arbitrary finite systems of ideal bosons, the discrete derivative of the function π_k yields the probabilities for condensing k bosons [4]. This relation between the cycle statistics and the condensate fraction has not been scrutinized before in detail. It allows us to determine the π_k from the known fluctuation properties of the ideal Bose gas. It is also possible, and very instructive, to compute the cycle probabilities π_k directly via the infinite-density limit of a finite Bose gas (see Sec. V). Furthermore, we will exploit some of the subtleties of the concept of cycle probabilities to compute the probability distribution of the maximum-length cycle at zero temperature, where the problem reduces to a study of random permutations [5]. Our preceding analysis of the π_k allows us to understand why this distribution is essentially unchanged at finite temperature, below the condensation temperature (Sec. VI). Finally, we briefly review some known relations between Bose condensation and the presence of infinite cycles for interacting Bose gases (Sec. VII).

II. RECURSION RELATIONS FOR THE PARTITION FUNCTION AND THE DENSITY MATRIX

The Landsberg recursion relation [6] gives Z_N as a sum of only N terms,

$$Z_N = \frac{1}{N} \sum_{k=1}^N Z_{N-k} z_k, \quad (5)$$

where the $z_k = Z_1(k\beta) = \sum_{\sigma} \exp(-k\beta\epsilon_{\sigma})$ are the single-particle partition functions at inverse temperature $k\beta$. In fact, the

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terms appearing in Eq. (5) are the cycle probabilities:

$$\pi_k = Z_{N-k} z_k / (N Z_N). \quad (6)$$

This is because k particles on a permutation cycle of length k at inverse temperature β have the same statistical weight z_k as a single particle at inverse temperature $k\beta$ and, furthermore, in the ideal Bose gas, the different cycles of a permutation are statistically independent. Thus in Eq. (6), the cycle contributes z_k , and the remaining $N-k$ particles contribute Z_{N-k} (see Ref. [7]).

The off-diagonal single-particle density matrix,

$$\rho_N(r, r', \beta) = \int dx_1 \cdots dx_{N-1} \times \rho^{\text{bos}}[\{x_1, \dots, x_{N-1}, r\}, \{x_1, \dots, x_{N-1}, r'\}, \beta],$$

is also a sum of permutation-dependent terms, and the Landsberg recursion relation can be generalized to a nondiagonal single-particle density matrix $\rho(r, r', k\beta)$ rather than to the diagonal one:

$$\rho_N(r, r', \beta) = \frac{1}{N} \sum_{k=1}^N Z_{N-k} \rho(r, r', k\beta), \quad (7)$$

$$\frac{V}{Z_N} \rho_N(r, r', \beta) = \sum_{k=1}^N \pi_k R_k^{\text{cut}}(r, r'), \quad (8)$$

where V is the volume of the system. In Eq. (8), the cycle probabilities π_k are modified by the cutoff function $R_k^{\text{cut}}(r, r')$, which is proportional to the ratio of the off-diagonal and the diagonal density matrices. In the three-dimensional homogeneous Bose gas in a periodic cubic box of size L^3 , we have $R_k^{\text{cut}}(r, r') = L^3 \rho(r, r', k\beta) / z_k = \rho(0, r-r', k\beta) / \rho(0, 0, k\beta)$.

As is well known, the condensate fraction is proportional to the off-diagonal single-particle density matrix in the limit $|r-r'| \rightarrow \infty$, a case that corresponds to $|r-r'| \propto L$ in a cubic box of length L for $L \rightarrow \infty$. The cutoff function then vanishes for small k , and terms with $k \gg \frac{L^2}{2\beta}$ dominate the sum in Eq. (8) [8].

III. OFF-DIAGONAL LONG-RANGE ORDER AND CYCLE PROBABILITIES

In this section, we discuss a relation between the off-diagonal long-range order parameter and the sum of cycle probabilities for the homogeneous Bose gas, with unit mass and particle density [$m = \hbar = 1$, $N = L^3$, $T_c = 2\pi / \zeta(3/2)^{2/3}$]. To analyze Eq. (8), we notice that $z_k \sim 1$ for $k \gg \frac{L^2}{2\beta}$ and $z_k \sim L^3 / (2\pi\beta k)^{3/2}$ for $k \ll \frac{L^2}{2\beta}$. It follows from $Z_{N-k} < Z_N$ that

$$\pi_k \leq \begin{cases} 1 / (2\pi\beta k)^{3/2} & \text{for } k \ll \frac{L^2}{2\beta} \\ 1/N & \text{for } k \gg \frac{L^2}{2\beta}. \end{cases} \quad (9)$$

We now study the cutoff function $R_k^{\text{cut}}(r, r')$ for $|r-r'| \propto L$. For concreteness, we suppose that the vector $r-r'$ has

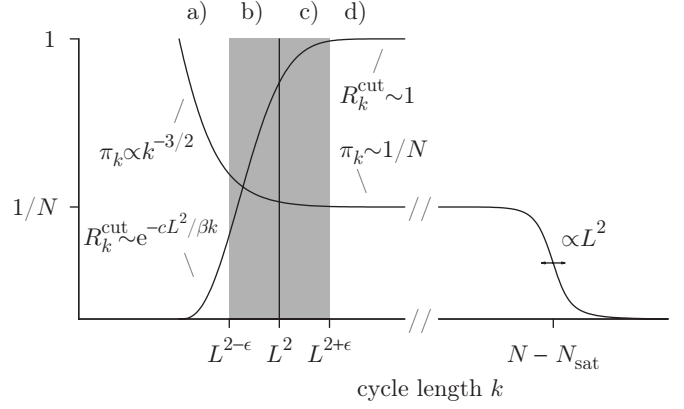


FIG. 1. Cycle probabilities π_k and cutoff function R_k^{cut} for a periodic box of size L^3 below T_c .

the same components in all three space directions: $r-r' = L(\delta, \delta, \delta)$ with $0 < \delta < \frac{1}{2}$ and denote the corresponding cutoff function and the off-diagonal density matrix by $R_k^{\text{cut}}(\delta)$ and by $\rho_N(\delta)$, respectively. We have

$$R_k^{\text{cut}}(\delta) = \left[\frac{\sum_{w \in \mathbb{Z}} e^{-(\delta+w)^2 L^2 / (2k\beta)}}{\sum_{w \in \mathbb{Z}} e^{-w^2 L^2 / (2k\beta)}} \right]^3.$$

For $k \gg \frac{L^2}{2\beta}$, both the numerator and the denominator are $\sim \sqrt{2\pi\beta k} / L$, so that $R_k^{\text{cut}}(\delta) \sim 1$. For $k \ll \frac{L^2}{2\beta}$, the numerator is $\sim \exp\{-3\delta^2 L^2 / (2\beta k)\}$, and the denominator is ~ 1 . This shows that the monotonic function $R_k^{\text{cut}}(\delta)$ is exponentially small for $k \ll \frac{L^2}{2\beta}$ and equal to unity for $k \gg \frac{L^2}{2\beta}$ (see Fig. 1). The estimates for R_k^{cut} and for π_k imply

$$\lim_{L \rightarrow \infty} \sum_{k=1}^{L^2} \pi_k R_k^{\text{cut}}(\delta) = 0,$$

$$\lim_{L \rightarrow \infty} \sum_{k=L^2}^N \pi_k [1 - R_k^{\text{cut}}(\delta)] = 0. \quad (10)$$

(see the Appendix for a short, but rigorous derivation). It follows that

$$\lim_{L \rightarrow \infty} \left[\frac{L^3 \rho_N(\delta)}{Z_N} - \sum_{k=\text{const} L^2}^N \pi_k \right] = 0. \quad (11)$$

This equation relates the probabilities of infinite cycles to off-diagonal long-range order and [because $\lim_{|r-r'| \rightarrow \infty} \frac{V}{Z_N} \rho_N(r, r', \beta)$ gives the condensate fraction] also to the condensate fraction. The relation between cycle lengths and off-diagonal long-range order is natural because, in order for a cycle to contribute to the off-diagonal density matrix for $|r-r'| \propto L$, its de Broglie wavelength must be at least comparable to the distance L , which means that the particle must belong to a cycle of length k with $L \lesssim \sqrt{2\pi k\beta}$ (see also Refs. [2,9] for related derivations).

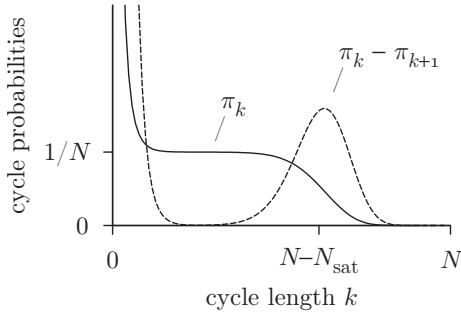


FIG. 2. Cycle probabilities π_k and their negative discrete derivative (not to scale) for $N=125$ ideal bosons in a periodic box of size $L^3=5^3$ at temperature $T/T_c=0.5$ ($N_{\text{sat}}=48.7$).

IV. CONDENSATE FRACTION AS A DERIVATIVE OF THE CYCLE PROBABILITIES

In Sec. III, we studied the sum of the cycle probabilities. We now discuss the fact that the discrete derivative of the cycle probabilities exactly gives the condensation probabilities. This relation allows us to compute the π_k , and it involves neither a cutoff function nor the thermodynamic limit and holds for arbitrary systems of ideal bosons. The relation was mentioned very briefly in the context of Monte Carlo calculations [4], but it was not analyzed in any detail.

Let us define $S_N(k, \sigma)$ as the canonic partition function with N bosons of which exactly k are in state σ (see Ref. [6]) and, similarly, the restricted partition function with at least k bosons in the state σ , as

$$Y_N(k, \sigma) = \sum_{k'=k}^N S_N(k', \sigma) = e^{-\beta k \epsilon_\sigma} Z_{N-k}. \quad (12)$$

[$S_N(k, \sigma)/Z_N$ is the probability of having exactly k bosons in state σ .] Summing Eq. (12) over all states σ , and dividing by NZ_N , we recover the cycle probabilities,

$$\frac{1}{NZ_N} \sum_{\sigma} Y_N(k, \sigma) = \frac{\sum_{\sigma} e^{-\beta k \epsilon_\sigma}}{NZ_N} Z_{N-k} = \pi_k,$$

whose negative discrete derivative,

$$\pi_k - \pi_{k+1} = \sum_{\sigma} S_N(k, \sigma)/(NZ_N), \quad (13)$$

thus yields the sum over all σ of the probabilities of condensing k bosons into state σ . We note that the left-hand side of Eq. (13) contains only quantities related to the path-integral picture, whereas the right-hand side involves only energy levels. Furthermore, we note that the probability of condensing k particles into excited levels is zero for large k , so that Eq. (13) effectively relates the ground-state condensation probabilities to the derivative of the cycle probabilities.

To illustrate the relation between cycles and condensation probabilities, we consider the example of Fig. 2 of N particles in a periodic box, with the cycle probabilities explicitly computed from the Landsberg recursion relation. For k

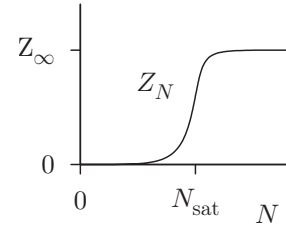


FIG. 3. Canonic partition function Z_N as a function of N in a finite system.

$> N/2$, the derivative of the cycle probabilities equals the sum of the probabilities of condensing k bosons into state σ . More generally, for $k \gg \frac{L^2}{2\beta}$, the condensation probabilities into the excited states vanish and $\pi_k - \pi_{k+1} \sim S_N(k, 0)/(NZ_N)$ directly yields the probability distribution of having k particles in the ground state, a distribution with mean value $k = \langle N_0 \rangle$ and standard deviation $\propto L^2$. On the other hand, for small k , as already discussed, the discrete derivative behaves as $k^{-5/2}$. It describes the probabilities of condensing k bosons into excited states. Finally, for $\frac{L^2}{2\beta} \ll k$ and $k \leq \langle N_0 \rangle - L^2/(2\pi\beta)^{3/2}$, the probability of having k particles in the condensate vanishes, so that the π_k are constant.

V. DIRECT CALCULATION OF CYCLE PROBABILITIES

We have seen in Sec. IV that the cycle probabilities π_k follow from the known fluctuations of the condensate in the canonic ensemble. It is very instructive to compute them directly via the infinite-density limit for the partition function of an arbitrary system. For $k \gg \frac{L^2}{2\beta}$, we have $z_k \sim 1$, so that $\pi_k \sim Z_{N-k}/Z_N$. For illustration, the Fig. 3 shows the Z_k in a fixed physical system (for the homogeneous Bose gas, in a fixed box of size L^3), as it can be computed numerically from the Landsberg recursion relation. The limiting value Z_∞ (at infinite density) and the value $k=N_{\text{sat}}$ of largest variation of Z_k can be computed exactly.

Indeed, as the ground-state energy is zero, Z_N differs from Z_{N-1} only by configurations without any particle in the ground state: $Z_N - Z_{N-1} = S_N(0, 0)$ [see above Eq. (12)]. Therefore $Z_N = \sum_{k=1}^N S_k(0, 0)$ with

$$S_k(0, 0) = \left[\prod_{\sigma>0} \sum_{n_\sigma=0}^{\infty} \right] \exp\left(-\beta \sum_{\sigma>0} n_\sigma \epsilon_\sigma\right) \delta_{\sum_{\sigma>0} n_\sigma, k}.$$

Summing this equation over all integers k yields, for arbitrary states $\{\epsilon_\sigma\}$,

$$Z_N = \prod_{\sigma>0} \frac{1}{1 - \exp(-\beta \epsilon_\sigma)} = \exp\left(\sum_{k=1}^{\infty} \frac{z_k - 1}{k}\right). \quad (14)$$

This exact formula for a finite canonical Bose system can be obtained without the usual saddle-point integration. It agrees with the grand-canonical partition function for the excited states at zero chemical potential because at high density the condensate serves as a reservoir for the excited bosons.

To determine the value of N for which the partition function Z_k passes from $Z_k \sim 0$ to $Z_k \sim Z_\infty$ (see Fig. 3), we again

take a discrete derivative and consider the probability distribution of the number of excited particles,

$$\pi_\infty(N_{\text{exc}} = k) = (Z_k - Z_{k-1})/Z_\infty,$$

whose mean value, the saturation number, is

$$\begin{aligned} N_{\text{sat}} = \langle N_{\text{exc}} \rangle &= \sum_{k=1}^{\infty} k \pi_\infty(N_{\text{exc}} = k) = \sum_{\sigma>0} \frac{\exp(-\beta \epsilon_\sigma)}{1 - \exp(-\beta \epsilon_\sigma)} \\ &= \sum_{k=1}^{\infty} (z_k - 1). \end{aligned} \quad (15)$$

The variance of the number of excited particles is again given exactly by the corresponding grand-canonical expressions

$$\begin{aligned} \langle N_{\text{exc}}^2 \rangle - \langle N_{\text{exc}} \rangle^2 &= \sum_{\sigma>0} \frac{1}{\exp(\beta \epsilon_\sigma) - 1} + \frac{1}{[\exp(\beta \epsilon_\sigma) - 1]^2} \\ &= N_{\text{sat}} + \sum_{k=2}^{\infty} (k-1)(z_k - 1). \end{aligned} \quad (16)$$

For completeness, we give the expressions corresponding to Eqs. (14)–(16) in the homogeneous Bose gas, in the limit $L \rightarrow \infty$: $Z_\infty \sim \exp[L^3/(2\pi\beta)^{-5/2}\zeta(5/2)]$ and, furthermore, $N_{\text{sat}}/L^3 \sim \zeta(3/2)/(2\pi\beta)^{3/2} = \eta_c$, with η_c the critical density. In addition, the variance of N_{exc} behaves as $L^4/(2\pi\beta)^3$ [10], so that the jump of Z_k/Z_N from 0 to 1, for $T < T_c$, takes place in a window of width L^2 , as already noted in Sec. IV.

Putting together the analysis of Z_k for a fixed size of the system and a varying particle number, and our knowledge of z_k , we have a complete description of the cycle probabilities π_k . Once we know exactly the behavior of the π_k in the thermodynamic limit, we observe that the sum of the π_k for $k \leq \frac{L^2}{2\beta}$ gives exactly the critical density η_c , and the sum of the π_k for $k \geq \frac{L^2}{2\beta}$ gives the condensate fraction. This conclusion is the same as in Sec. III, but it stems from a microscopic analysis of the cycle probabilities.

VI. MAXIMUM CYCLE LENGTHS

At finite temperature, the cycle probabilities π_k are easily computed from the Landsberg recursion relation, and at $T=0$, they are given by $\pi_k = 1/N$. In Monte Carlo calculations the entire permutation of N elements can be sampled, starting from the cycle containing the element N , which has length k with probability π_k . This leaves one with a system of $N-k$ particles, for which the probabilities of the cycle containing element $(N-k)$ can again be computed, etc. (see Ref. [7]). [At zero temperature, the distribution remains $\pi_k = 1/(N-k)$, at finite temperature, it is given by Eq. (6).]

In this section, we use the cycle probabilities π_k to obtain analytical results for p_k^{max} , the probability that the longest cycle in a permutation has length k . This probability distribution is closely related to the distribution π_k both at finite temperature and at $T=0$, where the ideal Bose gas is equivalent to the problem of random permutations, which has been much studied in the mathematics literature. We recover some

classic results which we generalize to finite temperatures.

We first note that the cycle probabilities are related to the mean number of cycles of length k , $\tilde{\pi}_k = N\pi_k/k$. For $k > N/2$, $\tilde{\pi}_k$ coincides with the probability p_k that the permutation P has at least one cycle of length k . This is simply because a permutation of N elements can have no more than one such cycle, in other words, because

$$\tilde{\pi}_k = \sum_{m=1}^{\infty} m \times \left\{ \begin{array}{l} \text{prob. to have} \\ m \text{ cycles of length } k \end{array} \right\} \quad (17)$$

and

$$p_k = \sum_{m=1}^{\infty} \left\{ \begin{array}{l} \text{prob. to have} \\ m \text{ cycles of length } k \end{array} \right\} \quad (18)$$

and only terms with $m=1$ contribute to the above expressions if $k > N/2$.

Furthermore, any cycle of length $k > N/2$ must be the longest cycle, so that we arrive for all temperatures at

$$p_k^{\text{max}} = p_k = \tilde{\pi}_k = N\pi_k/k \quad \text{for } k > N/2, \quad \forall T. \quad (19)$$

We next consider the probability distributions p_k and p_k^{max} for $k \leq N/2$ at $T=0$, for $N \rightarrow \infty$. We take the continuum limit $p_k \rightarrow p(x)$ by setting $p_k = \frac{1}{N}p(x=k/N)$ [and similarly for $p_k^{\text{max}}(x)$ and $\tilde{\pi}(x)$].

It follows from the recursive procedure, and from the fact that the π_k are of order $1/N$, that the probability to have more than one cycle of the same length is $O(1/N^2)$. In the limit $N \rightarrow \infty$, we see that $p(x) = \tilde{\pi}(x) = 1/x$.

For $x < 1/2$, the probability $p^{\text{max}}(x)$ is the product of the probability to sample x , and the probability that x remains the longest cycle in the remaining partition of length $1-x$, in other words,

$$p^{\text{max}}(x) = p(x) \left(1 - \left\{ \begin{array}{l} \text{prob. that longest cycle in} \\ [0, 1-x] \text{ is } > x \end{array} \right\} \right)$$

At $T=0$, and in the limit $N \rightarrow \infty$, the probability that in a permutation of length $N(1-x)$ the longest cycle's length exceeds Nx equals the probability that in a permutation of N elements the longest cycle exceeds $x/(1-x)$, and we arrive at

$$p^{\text{max}}(x) = p(x) \left[1 - \int_{x/(1-x)}^1 dx' p^{\text{max}}(x') \right]. \quad (20)$$

For all $x \in]0, 1[$, $x < x/(1-x)$, so Eq. (20) can be used to compute $p^{\text{max}}(x)$ from $x=1$ downwards to arbitrary precision (see Fig. 4). Alternatively, we can transform Eq. (20) into a differential equation

$$\frac{d}{dx} p^{\text{max}}(x) = -\frac{1}{x} p^{\text{max}}(x) + \frac{1}{x(1-x)^2} p^{\text{max}}\left(\frac{x}{1-x}\right).$$

Explicit formulas for $p^{\text{max}}(x)$ are easily obtained, starting by inputting, from Eq. (19), $p^{\text{max}}(x)$ for $\frac{1}{2} < x < 1$ to obtain p^{max} in the window $x \in [\frac{1}{3}, \frac{1}{2}]$, etc.,

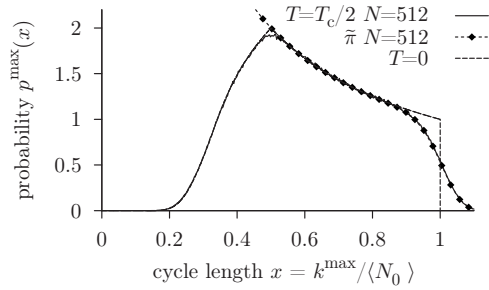


FIG. 4. Probability $p^{\max}(x)$ from a numerical integration of Eq. (20) at $T=0$, compared to the distribution p_k^{\max} obtained from a Monte Carlo sampling at $T=T_c/2$, and the exact expression of Eq. (19) for large x , at $T=T_c/2$.

$$p^{\max}(x) = \begin{cases} \frac{1}{x} & \text{for } \frac{1}{2} < x \\ \frac{1}{x} \left[1 - \ln\left(\frac{1-x}{x}\right) \right] & \text{for } \frac{1}{3} < x < \frac{1}{2}, \end{cases}$$

but they become cumbersome (see also Ref. [11]).

At finite temperature, the cycle of the element N is sampled from a distribution π_k as shown in Fig. 2 rather than from the flat distribution. However, the longest cycle is again of length $\propto N$, and it is sampled from an essentially constant distribution in the interval $x \in [0, \langle N_0 \rangle / N]$. In the thermodynamic limit, for $T < T_c$, the distribution $p^{\max}(x)$ of the properly rescaled variable $x = k^{\max} / \langle N_0 \rangle$ trivially agrees with the one obtained at $T=0$. In particular, the mean length of the longest cycle is given by $0.624 \langle N_0 \rangle$ at all temperatures $T < T_c$ [5]. At sufficiently low temperature, the rounding at $x \approx 1$ is *exactly* described by Eq. (19) (see Fig. 4).

VII. INTERACTING BOSE GAS

For the interacting Bose gas, the relation between off-diagonal long-range order and the infinite cycles has been studied for generalized mean-field models ([2,12]), where it was shown that Bose-Einstein condensation and finite density of particles belonging to infinite cycles are equivalent. But it has not been possible to establish analogous rigorous results for models with realistic interactions. In the general case, the definition of the superfluid density in terms of the winding numbers [13] proves that for the homogeneous Bose gas in three dimensions, the presence of infinite cycles is equivalent to a nonzero superfluid density. It is generally admitted that in homogeneous systems, Bose-Einstein condensation implies superfluidity, so that the presence of a condensate would imply the existence of infinite cycles in those systems.

On the other hand, the question whether superfluidity implies Bose-Einstein condensation for homogeneous bosonic quantum systems (above two dimensions) has no rigorous answer. (We note that two-dimensional superfluids are not Bose condensed.) In three dimensions, empirical evidence points to the direct link between superfluidity in homogeneous superfluid systems and Bose-Einstein condensation, but a proof that superfluidity implies Bose-Einstein conden-

sation is still lacking. This is related to the curious difficulty, pointed out for example in Ref. [14], to rigorously prove the existence of Bose-Einstein condensation for interacting systems.

In any case, the direct link between the cycle probabilities and the condensate fraction, as described in Sec. IV, cannot hold for interacting systems. This is easily seen at $T=0$ because the ground state of any system of identical but distinguishable particles has bosonic symmetry. Therefore the distinguishable-particle density matrix

$$\rho[\{x_1, \dots, x_N\}, \{x_{P_1}, \dots, x_{P_N}\}, \beta = \infty]$$

is independent of the permutation P . It follows that at $T=0$, and for finite N , all permutations have the same weight and the cycle probabilities again satisfy $\pi_k = \frac{1}{N}$. At the same time, the condensate fraction of interacting systems at $T=0$ is less than 1 [15], and the distribution of the π_k cannot be related to the distribution of N_0 in the same manner as in the ideal Bose gas.

VIII. CONCLUSION

In conclusion, we have discussed the relation between the off-diagonal long-range order, the condensate fraction, and the cycle probabilities in the ideal Bose gas. Only long cycles ($k > \text{const } L^2$ in the homogeneous case) contribute to the off-diagonal long-range order parameter, and Bose-Einstein condensation is equivalent to the presence of infinite cycles. We have also discussed the probability distribution for long cycles, and the general, nonintuitive, link between the integer derivative of the cycle probabilities and the number of condensed bosons. This integer derivative provides us with a purely topological characterization of the condensate fraction in the ideal Bose gas. Our knowledge of the cycle probabilities allows us to study the distribution of the longest cycle. We have remarked that the $0 < T < T_c$ case can be understood from the $T=0$ case. The mean length of the longest cycle is proportional to the number of condensed bosons. For the interacting Bose gas, at the present time, only the winding-number formula [13] has been rigorously shown to relate the topology of Feynman paths to another characteristic of interacting Bose-condensed systems, namely the superfluid density. The link between Bose-Einstein condensation and superfluidity is not clearly elucidated [14] and it will need to be explored further in order to better understand the cycles of the interacting Bose gas.

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APPENDIX: DERIVATION OF EQ. (10)

In this Appendix, we prove Eq. (10). We consider the sums separately in the regions (a)–(d) indicated in Fig. 1 and show that they all vanish individually.

In region (a), $\pi_k R_k^{\text{cut}}(\delta) \leq \pi_k \exp[-3\delta^2 L^\epsilon / \beta]$, and the normalization of the π_k shows that the partial sum vanishes in the limit $L \rightarrow \infty$. In (b), $\pi_k R_k^{\text{cut}} \leq \pi_{L^{2-\epsilon}} R_{L^2}^{\text{cut}} \leq \text{const}/L^{3(2-\epsilon)/2}$ and the partial sum is smaller than $\text{const} L^{3\epsilon/2-1}$. In (c), as $1 - R_k^{\text{cut}}(\delta)$ decreases with k , $\pi_k [1 - R_k^{\text{cut}}(\delta)] \leq \pi_{L^2} [1 - R_{L^2}^{\text{cut}}(\delta)]$ and the partial sum is dominated by $\text{const} L^{\epsilon-1}$. Finally, in

(d), $\pi_k [1 - R_k^{\text{cut}}(\delta)] \leq \pi_k [1 - R_{L^{2+\epsilon}}^{\text{cut}}(\delta)]$. The normalization condition of the π_k and the relation $L^{2+\epsilon} \gg L^2$ imply that this partial sum vanishes for $L \rightarrow \infty$. With a suitable choice of ϵ (for example $\epsilon = \frac{1}{2}$), these sums all vanish for $L \rightarrow \infty$. In fact, any function $\epsilon(L) \gg 1/\ln L$ that satisfies $\epsilon(L) < 2/3$ for $L > L_0$ can be used.

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